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Partitions of natural numbers with the same representation functions ☆

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ABSTRACT

For a set A of nonnegative integers the representation functions $R_2(A,n)$, $R_3(A,n)$ are defined as the number of solutions of the equation $n=a+a', a,a'\in A$ with $a<a'$, $a\leqslant a'$, respectively. Let $D(0)=0$ and let $D(a)$ denote the number of ones in the binary representation of a. Let A_0 be the set of all nonnegative integers a with even $D(a)$ and A_1 be the set of all nonnegative integers a with odd $D(a)$. In this paper we show that (a) if $R_2(A,n)=R_2(\mathbb{N}\setminus A,n)$ for all $n\geqslant 2N-1$, then $R_2(A,n)=R_2(\mathbb{N}\setminus A,n)\geqslant 1$ for all $n\geqslant 12N^2-10N-2$ except for $A=A_0$ or $A=A_1$; (b) if $R_3(A,n)=R_3(\mathbb{N}\setminus A,n)$ for all $n\geqslant 2N-1$, then $R_3(A,n)=R_3(\mathbb{N}\setminus A,n)\geqslant 1$ for all $n\geqslant 12N^2+2N$. Several problems are posed in this paper.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For a set $A\subseteq\mathbb{N}$, let $R_1(A,n)$, $R_2(A,n)$, $R_3(A,n)$ denote the number of solutions of

$$a+a'=n,\quad a,a'\in A,$$
$$a+a'=n,\quad a,a'\in A,\ a<a',$$
$$a+a'=n,\quad a,a'\in A,\ a\leqslant a',$$

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respectively. For $i \in \{1, 2, 3\}$, Sárközy asked ever whether there are sets A and B with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers n. Dombi [3] proved that the answer is negative for $i = 1$ and positive for $i = 2$. For $i = 3$, Chen and Wang (see [2]) proved that the set of nonnegative integers can be partitioned into two subsets A and B with $R_3(A, n) = R_3(B, n)$ for all $n \geqslant n_0$. In [7], Lev gave a simple common proof to the results by Dombi [3] and Chen and Wang [2]. In [10], using generating functions, Sándor proved the following precise formulations. Recently, Tang [11] gave a more natural proof of Sándor's results.

Theorem A. *Let N be a positive integer. Then $R_2(A, n) = R_2(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \in A$, $2m + 1 \in A \Leftrightarrow m \notin A$ for all $m \geqslant N$.*

Theorem B. *Let N be a positive integer. Then $R_3(A, n) = R_3(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \notin A$, $2m + 1 \in A \Leftrightarrow m \in A$ for all $m \geqslant N$.*

For other related results, see [1,4–6,8,9,12,13].

Let $D(0) = 0$ and let $D(a)$ denote the number of ones in the binary representation of a. Let A_0 be the set of all nonnegative integers a with even $D(a)$ and A_1 be the set of all nonnegative integers a with odd $D(a)$. Then, for any $N \geqslant 1$, both $A = A_0$ and $A = A_1$ satisfy the conditions that $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \in A$, $2m + 1 \in A \Leftrightarrow m \notin A$ for all $m \geqslant N$, which follow by induction on N.

In this paper, the following results are proved.

Theorem 1. *Let N be a positive integer such that $R_2(A, n) = R_2(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$. Then $R_2(A, n) = R_2(\mathbb{N} \setminus A, n) \geqslant 1$ for all $n \geqslant 12N^2 - 10N - 2$ except for $A = A_0$ or $A = A_1$.*

For $A = A_0$ or $A = A_1$ we have

$$R_2\big(A, 2^{2n+1} - 1\big) = R_2\big(\mathbb{N} \setminus A, 2^{2n+1} - 1\big) = 0, \quad \text{for all } n \geqslant 0.$$

Theorem 2. *Let N be a positive integer such that $R_3(A, n) = R_3(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$. Then $R_3(A, n) = R_3(\mathbb{N} \setminus A, n) \geqslant 1$ for all $n \geqslant 12N^2 + 2N$.*

Remark 1. Define $R_{A,B}(n)$ to be the number of solutions of $a + b = n$, $a \in A$, $b \in B$. Similarly to the proofs of Theorems 1 and 2, we can prove that (a) if $R_2(A, n) = R_2(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$, then $R_{A, \mathbb{N} \setminus A}(n) \geqslant 1$ for all $n \geqslant 12N^2 - 10N - 2$ except for $A = A_0$ or $A = A_1$; (b) if $R_3(A, n) = R_3(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N - 1$, then $R_{A, \mathbb{N} \setminus A}(n) \geqslant 1$ for all $n \geqslant 12N^2 + 2N$.

Remark 2. The Erdős–Turán Conjecture (see [9]) says that if the values of a representation function of a set are not zero for all sufficiently large n, then the representation function cannot be bounded. But the conjecture is still open. So it is natural to concern the cases in Theorems 1 and 2.

If A satisfies the condition in Theorem 1, then by Lemma 1 in Section 2 we can prove that

$$\limsup_{n \to +\infty} R_2(A, n) = +\infty.$$

Indeed, if we take a positive integer $m \in A$ with $m \geqslant N$ and $n = 2^{2t+1}m + 2^{2t} - 1$, then for any s with $0 \leqslant s \leqslant t - 1$, by Lemma 1, we have $2^{2t}m + 2^{2s} - 1 \in A$ and $2^{2t}m + 2^{2t} - 2^{2s} \in A$. Hence $R_2(A, n) \geqslant t$. Similarly, if A satisfies the condition in Theorem 2, then by Lemma 2 in Section 3 we can prove that

$$\limsup_{n \to +\infty} R_3(A, n) = +\infty.$$

Currently we have no answers for the following problems.

Problem 1. Are there any $A \subseteq \mathbb{N}$ with $R_2(A,n) = R_2(\mathbb{N} \setminus A, n)$ for all sufficiently large n such that

$$\liminf_{n\to+\infty} R_2(A,n) = +\infty?$$

Problem 2. Are there any $A \subseteq \mathbb{N}$ with $R_3(A,n) = R_3(\mathbb{N} \setminus A, n)$ for all sufficiently large n such that

$$\liminf_{n\to+\infty} R_3(A,n) = +\infty?$$

Problem 3. Are there any $A \subseteq \mathbb{N}$ with $R_2(A,n) = R_2(\mathbb{N} \setminus A, n)$ for all sufficiently large n such that

$$\liminf_{n\to+\infty} R_{A,\,\mathbb{N}\setminus A}(n) = +\infty?$$

Problem 4. Are there any $A \subseteq \mathbb{N}$ with $R_3(A,n) = R_3(\mathbb{N} \setminus A, n)$ for all sufficiently large n such that

$$\liminf_{n\to+\infty} R_{A,\,\mathbb{N}\setminus A}(n) = +\infty?$$

2. Proof of Theorem 1

In this section we always assume that $R_2(A,n) = R_2(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N-1$.

Lemma 1. *Let m, k, i be integers with $m \geqslant N$, $i \geqslant 0$ and $0 \leqslant k < 2^i$. Then*

(a) *if $2 \mid D(k)$, then $m \in A \Leftrightarrow 2^i m + k \in A$;*
(b) *if $2 \nmid D(k)$, then $m \in A \Leftrightarrow 2^i m + k \notin A$.*

Proof. We use induction on i. It is clear for $i = 0$. Suppose that Lemma 1 is true for all $0 \leqslant i < l$. Now we consider the case $i = l$.

Case 1. $2 \mid k$.

If $2 \mid D(k)$, then $2 \mid D(\frac{k}{2})$. By the inductive hypothesis and Theorem A we have

$$m \in A \quad\Leftrightarrow\quad 2^{l-1}m + \frac{k}{2} \in A \quad\Leftrightarrow\quad 2\left(2^{l-1}m + \frac{k}{2}\right) \in A \quad\Leftrightarrow\quad 2^l m + k \in A.$$

If $2 \nmid D(k)$, then $2 \nmid D(\frac{k}{2})$. By the inductive hypothesis and Theorem A we have

$$m \in A \quad\Leftrightarrow\quad 2^{l-1}m + \frac{k}{2} \notin A \quad\Leftrightarrow\quad 2\left(2^{l-1}m + \frac{k}{2}\right) \notin A \quad\Leftrightarrow\quad 2^l m + k \notin A.$$

Case 2. $2 \nmid k$.

If $2 \mid D(k)$, then $2 \nmid D(\frac{k-1}{2})$. By the inductive hypothesis and Theorem A we have

$$m \in A \quad\Leftrightarrow\quad 2^{l-1}m + \frac{k-1}{2} \notin A \quad\Leftrightarrow\quad 2(2^{l-1}m + \frac{k-1}{2}) + 1 \in A \quad\Leftrightarrow\quad 2^l m + k \in A.$$

If $2 \nmid D(k)$, then $2 \mid D(\frac{k-1}{2})$. By the inductive hypothesis and Theorem A we have

$$m \in A \quad\Leftrightarrow\quad 2^{l-1}m + \frac{k-1}{2} \in A \quad\Leftrightarrow\quad 2(2^{l-1}m + \frac{k-1}{2}) + 1 \notin A \quad\Leftrightarrow\quad 2^l m + k \notin A.$$

This completes the proof of Lemma 1. $\square$

Proof of Theorem 1. It is clear that $\mathbb{N} = A_0 \cup A_1$ and

$$R_2(A_0, 2^{2n+1} - 1) = R_2(A_1, 2^{2n+1} - 1) = 0, \quad \text{for all } n \geqslant 0.$$

Now we assume that $0 \in A$ and $A \neq A_0$, otherwise, replacing A by $\mathbb{N} \setminus A$. Then $(A_0 \setminus A) \cup (A \setminus A_0) \neq \emptyset$. Let a be the least nonnegative integer in $(A_0 \setminus A) \cup (A \setminus A_0)$. By the definition of A_0 and Theorem A we have $a \leqslant 2N - 1$, otherwise, $b \in A_0 \Leftrightarrow b \in A$ for all $b \leqslant 2N - 1$, by Theorem A we have $A = A_0$. If $N = 1$, then by $0 \in A$ and Theorem A we have $A = A_0$. So $N \geqslant 2$.

Suppose that there is an integer $n \geqslant 12N^2 - 10N - 2$ with

$$R_2(A, n) = R_2(\mathbb{N} \setminus A, n) = 0. \tag{1}$$

If $n = 3m$ with $m \in \mathbb{N}$, then

$$m \in A \overset{\text{Thm A}}{\Longleftrightarrow} 2m \in A \overset{(1)}{\Longleftrightarrow} m \notin A,$$

a contradiction.

If $n = 3m + 2$ with $m \in \mathbb{N}$, then

$$m \in A \overset{\text{Thm A}}{\Longleftrightarrow} 2m+1 \notin A \overset{(1)}{\Longleftrightarrow} m+1 \in A \overset{\text{Thm A}}{\Longleftrightarrow} 2m+2 \in A \overset{(1)}{\Longleftrightarrow} m \notin A,$$

a contradiction.

If $n = 6m + 4$ with $m \in \mathbb{N}$, then

$$\begin{aligned} m \in A &\overset{\text{Thm A}}{\Longleftrightarrow} 2m \in A \overset{(1)}{\Longleftrightarrow} 4m+4 \notin A \overset{\text{Thm A}}{\Longleftrightarrow} 2m+2 \notin A \\ &\overset{(1)}{\Longleftrightarrow} 4m+2 \in A \overset{\text{Thm A}}{\Longleftrightarrow} 2m+1 \in A \overset{\text{Thm A}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction.

So we may assume that $n \equiv 1 \pmod 6$. Write $n = 2^k t - 1$ with $2 \nmid t$.

Case 1. $2 \nmid k$. Then by $n \equiv 1 \pmod 6$ we have $t \equiv 1 \pmod 3$. Since $2 \nmid t$, we may write $t = 6m + 1$ with $m \in \mathbb{N}$. Thus $n = 2^{k+1}3m + 2^k - 1$. If $m \geqslant N + 1$, then

$$\begin{aligned} m \in A &\overset{\text{Lemma 1}}{\Longleftrightarrow} 2^{k+2}m + 2^k \notin A \overset{(1)}{\Longleftrightarrow} 2^{k+1}(m-1) + 2^{k+1} - 1 \in A \\ &\overset{\text{Lemma 1}}{\Longleftrightarrow} m - 1 \in A \overset{\text{Lemma 1}}{\Longleftrightarrow} 2^{k+1}(m-1) \in A \\ &\overset{(1)}{\Longleftrightarrow} 2^{k+2}m + 2^{k+1} + 2^k - 1 \notin A \overset{\text{Lemma 1}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction. Hence $m \leqslant N$ and $t \leqslant 6N + 1$.

Case 2. $2 \mid k$. Then by $n \equiv 1 \pmod 6$ we have $t \equiv 2 \pmod 3$. Since $2 \nmid t$, we may write $t = 6m + 5$ with $m \in \mathbb{N}$. Thus $n = 2^{k+1}3m + 2^k 5 - 1$. If $m \geqslant N$, then

$$\begin{aligned} m \in A &\overset{\text{Lemma 1}}{\Longleftrightarrow} 2^{k+1}m \in A \overset{(1)}{\Longleftrightarrow} 2^{k+2}(m+1) + 2^k - 1 \notin A \\ &\overset{\text{Lemma 1}}{\Longleftrightarrow} m + 1 \notin A \overset{\text{Lemma 1}}{\Longleftrightarrow} 2^{k+1}(m+1) \notin A \\ &\overset{(1)}{\Longleftrightarrow} 2^{k+2}m + 2^{k+1} + 2^k - 1 \in A \overset{\text{Lemma 1}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction. Hence $m \leqslant N - 1$ and $t \leqslant 6N - 1$.

In each case we have $t \leqslant 6N+1$. Let k_1 be the least nonnegative integer with $2^{k_1}t \geqslant N+1$. Let $2^{k_1}t = M+1$ and $k-k_1 = k_2$. Then $N \leqslant M \leqslant 6N$ and $n = 2^{k_2}M + 2^{k_2} - 1$. By $n \geqslant 12N^2 - 10N - 2$ and $M \leqslant 6N$ we have $2^{k_2} - 1 \geqslant 2N - 1 \geqslant a$. By the assumption $0 \in A$ and (1) we have

$$2^{k_2}M + 2^{k_2} - 1 \notin A.$$

By Lemma 1 we have $M \in A \Leftrightarrow 2 \nmid k_2$.

If $a \in A_0 \setminus A$, then by (1) we have

$$2^{k_2}M + 2^{k_2} - 1 - a \in A.$$

By Lemma 1 we have $M \in A \Leftrightarrow 2 \mid k_2 - D(a) \Leftrightarrow 2 \mid k_2$, a contradiction.

If $a \in A \setminus A_0$, then by (1) we have

$$2^{k_2}M + 2^{k_2} - 1 - a \notin A.$$

By Lemma 1 we have $M \in A \Leftrightarrow 2 \nmid k_2 - D(a) \Leftrightarrow 2 \mid k_2$, a contradiction.

This completes the proof of Theorem 1. □

3. Proof of Theorem 2

In this section we always assume that $R_3(A,n) = R_3(\mathbb{N} \setminus A, n)$ for all $n \geqslant 2N-1$.

Lemma 2. *Let m, k, i be integers with $m \geqslant N$, $i \geqslant 0$ and $0 \leqslant k < 2^i$. Then*

(a) *if $2 \mid i + D(k)$, then $m \in A \Leftrightarrow 2^i m + k \in A$;*
(b) *if $2 \nmid i + D(k)$, then $m \in A \Leftrightarrow 2^i m + k \notin A$.*

A proof is similar to Lemma 1.

Proof of Theorem 2. Without loss of generality, we may assume that $0 \in A$. If $N \notin (A_0 \setminus A) \cup (A \setminus A_0)$, then either $N \in A_0 \cap A$ or $N \notin A_0 \cup A$. If $N \in A_0 \cap A$, then by the definition of A_0 and Theorem B we have $2N \in A_0$ and $2N \notin A$. If $N \notin A_0 \cup A$, then by the definition of A_0 and Theorem B we have $2N \notin A_0$ and $2N \in A$. So $(A_0 \setminus A) \cup (A \setminus A_0) \neq \emptyset$. Let a be the least nonnegative integer in $(A_0 \setminus A) \cup (A \setminus A_0)$. By the above arguments we have $a \leqslant 2N$.

Suppose that there is an integer $n \geqslant 12N^2 + 2N$ with

$$R_3(A,n) = R_3(\mathbb{N} \setminus A, n) = 0. \tag{2}$$

If $n = 3m+1$ with $m \in \mathbb{N}$, then

$$m \in A \quad \overset{\text{Thm B}}{\Longleftrightarrow} \quad 2m+1 \in A \quad \overset{(2)}{\Longleftrightarrow} \quad m \notin A,$$

a contradiction.

If $n = 3m+2$ with $m \in \mathbb{N}$, then

$$m \in A \quad \overset{\text{Thm B}}{\Longleftrightarrow} \quad 2m+1 \in A \quad \overset{(2)}{\Longleftrightarrow} \quad m+1 \notin A \quad \overset{\text{Thm B}}{\Longleftrightarrow} \quad 2m+2 \in A \quad \overset{(2)}{\Longleftrightarrow} \quad m \notin A,$$

a contradiction.

If $n = 6m$ with $m \in \mathbb{N}$, then

$$\begin{aligned} m \in A &\overset{\text{Thm B}}{\Longleftrightarrow} 2m+1 \in A \overset{(2)}{\Longleftrightarrow} 4m-1 \notin A \overset{\text{Thm B}}{\Longleftrightarrow} 2m-1 \notin A \\ &\overset{(2)}{\Longleftrightarrow} 4m+1 \in A \overset{\text{Thm B}}{\Longleftrightarrow} 2m \in A \overset{\text{Thm B}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction.

So we may assume that $n \equiv 3 \pmod 6$. Write $n = 2^k t - 1$ with $2 \nmid t$.

Case 1. $2 \mid k$. Then by $n \equiv 3 \pmod 6$ we have $t \equiv 1 \pmod 3$. Since $2 \nmid t$, we may write $t = 6m+1$ with $m \in \mathbb{N}$. Thus $n = 2^{k+1}3m + 2^k - 1$. If $m \geqslant N+1$, then

$$\begin{aligned} m \in A &\overset{\text{Lemma 2}}{\Longleftrightarrow} 2^{k+2}m + 2^k \notin A \overset{(2)}{\Longleftrightarrow} 2^{k+1}(m-1) + 2^{k+1} - 1 \in A \\ &\overset{\text{Lemma 2}}{\Longleftrightarrow} m-1 \in A \overset{\text{Lemma 2}}{\Longleftrightarrow} 2^{k+1}(m-1) \notin A \\ &\overset{(2)}{\Longleftrightarrow} 2^{k+2}m + 2^{k+1} + 2^k - 1 \in A \overset{\text{Lemma 2}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction. Hence $m \leqslant N$ and $t \leqslant 6N+1$.

Case 2. $2 \nmid k$. Then by $n \equiv 3 \pmod 6$ we have $t \equiv 2 \pmod 3$. Since $2 \nmid t$, we may write $t = 6m+5$ with $m \in \mathbb{N}$. Thus $n = 2^{k+1}3m + 2^k 5 - 1$. If $m \geqslant N$, then

$$\begin{aligned} m \in A &\overset{\text{Lemma 2}}{\Longleftrightarrow} 2^{k+1}m \in A \overset{(2)}{\Longleftrightarrow} 2^{k+2}(m+1) + 2^k - 1 \notin A \\ &\overset{\text{Lemma 2}}{\Longleftrightarrow} m+1 \notin A \overset{\text{Lemma 2}}{\Longleftrightarrow} 2^{k+1}(m+1) \notin A \\ &\overset{(2)}{\Longleftrightarrow} 2^{k+2}m + 2^{k+1} + 2^k - 1 \in A \overset{\text{Lemma 2}}{\Longleftrightarrow} m \notin A, \end{aligned}$$

a contradiction. Hence $m \leqslant N-1$ and $t \leqslant 6N-1$.

In each case we have $t \leqslant 6N+1$. Let k_1 be the least nonnegative integer with $2^{k_1}t \geqslant N+1$. Let $2^{k_1}t = M+1$ and $k - k_1 = k_2$. Then $N \leqslant M \leqslant 6N$ and $n = 2^{k_2}M + 2^{k_2} - 1$. By $n \geqslant 12N^2 + 2N$ and $M \leqslant 6N$ we have $2^{k_2} - 1 \geqslant 2N \geqslant a$. By the assumption $0 \in A$ and (2) we have

$$2^{k_2}M + 2^{k_2} - 1 \notin A.$$

By Lemma 2 and $M \geqslant N$ we have $M \notin A$.

If $a \in A_0 \setminus A$, then by (2) we have

$$2^{k_2}M + 2^{k_2} - 1 - a \in A.$$

By Lemma 2, $M \notin A$ and $M \geqslant N$ we have $2 \nmid 2k_2 - D(a)$, a contradiction with $a \in A_0$.

If $a \in A \setminus A_0$, then by (2) we have

$$2^{k_2}M + 2^{k_2} - 1 - a \notin A.$$

By Lemma 2, $M \notin A$ and $M \geqslant N$ we have $2 \mid 2k_2 - D(a)$, a contradiction with $a \notin A_0$. This completes the proof of Theorem 2. □

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